OBSERVABILITY AND REDUNDANCY CLASSIFICATION
IN PROCESS NETWORKS
THEOREMS AND ALGORITHMS

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Abstract—The utility of observability and redundancy in characterizing the performance of process data estimators was established in previous studies[10]. In this paper two classification algorithms for determining local and global observability and redundancy for individual variables and measurements are presented. The concepts of biconnected components, perturbation subgraphs and feasible unmeasurable perturbations are introduced, and their properties are developed and used to effect classification, simplification and dimensional reduction. Step-by-step application of these algorithms is illustrated by examples.

INTRODUCTION

In a previous paper[10] we developed the concepts of observability and redundancy for constrained steady state systems and demonstrated that these concepts are useful in characterizing the performance of process data estimators with regard to bias and uniqueness of an estimate, convergence of estimation procedures and the feasibility and implications of problem decomposition. We also derived first and second order sufficient conditions for local observability and, in the case of linear constraints and measurements, global observability. However, from the practical point of view the application of these conditions suffers two serious drawbacks. In the first place, they are based on matrix rank tests which are computationally cumbersome. For systems for which local observability classification is of no concern, one approach might be to perform a "structural" rank test[8]. Secondly, and even more seriously, since these tests are applied to the system as a whole, they cannot be used to classify the observability of individual variables. In this paper we shall show that by exploiting the structural characteristics of a process network we can develop theorems and algorithms which would classify individual variables in such a network.

There has been a growing interest recently in developing graph-theoretical controllability criteria, which are closely related to observability criteria, for linear dynamic systems. Göknar[3, 4] developed necessary and sufficient conditions, and an algorithm for testing controllability using signal-flow graphs. Lin[5] introduced the notion of "structural controllability", which is independent of the values of non-zero system parameters. This concept was related to properties of a directed graph derived from the structure of the system equations. In this paper we shall develop graph-theoretical observability and redundancy criteria for steady state process networks.

PROCESS NETWORK AND PERTURBATION SUBGRAPHS

We shall begin with a discussion of some graph-theoretic concepts and terminology needed for the classification algorithms, but it is not our intention to provide a comprehensive introduction to graph theory. The reader is referred to Mah and Shacham[7] for a summary of some very useful properties of graphs and digraphs and to Deo[2] for a fuller treatment of graph theory.

As we have shown in a previous paper[6], the process graph is a very useful representation of the topological structure of a process. We used that representation to obtain certain decomposition results for process data reconciliation, coaption and fault detection. In this investigation we shall further exploit the structural properties of such a representation. We shall start with the properties of the underlying graph in which the directions of arcs have been erased.

A node v is a cut-node or articulation point of a connected graph G if its removal disconnects G. For instance, in Fig. 1(a) nodes 2 and 5 are cut-nodes. A graph is separable or 1-connected if it contains a cut-node. Now suppose we split a cut-node into two nodes to produce two disjoint subgraphs and let us refer to this operation as splitting. If we repeat this operation until all subgraphs are non-separable, then the resulting subgraphs are called blocks or biconnected components. The four biconnected components derived from the graph in Fig. 1(a) are shown in Fig. 1(b).

Clearly, two arcs belong to the same biconnected component if and only if they belong to a common cycle. Any two biconnected components are either disjoint or have exactly one cut-node in common. Each cut-node lies in at least two biconnected components and all other nodes can belong to only one biconnected component. It can also be shown that the rank and nullity of a graph are preserved in the splitting operation. Further discussion

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of the properties of biconnected components is given elsewhere [11, 1].

Another operation which will be carried out in the classification algorithm is the aggregation of two adjacent nodes with the elimination of all arcs between them. We shall refer to this operation simply as aggregation and the aggregated node as a pseudo-node.

By contrast to a process graph we shall use the term "process network" to refer to both the structure of the graph and the attributes of the arcs. Schematically, solid lines and broken lines are used to represent mass flow arcs and pure energy flow arcs, and temperature and flow measurements are indicated by single and double slashes on the respective arcs, as shown in Fig. 2. On the basis of these attributes we can define a hierarchy of subgraphs. $G_s$ is the subgraph of $G$ with all measured pure energy flow arcs deleted. $G_m$ is the subgraph of $G$ with all pure energy flow arcs deleted. $G_{m_u}$ is obtained from $G_m$ by deleting all arcs with flow measurements and $G_{m_u}$ is obtained from $G_{m_u}$ by deleting all arcs with

![Graph and its biconnected components](image)

**Fig. 1.** A graph and its biconnected components.

![Process network and subgraphs](image)

**Fig. 2.** A process network and some of its subgraphs: (a) a process network; (b) completely unmeasured mass flow subgraph $G_{m_u}$; (c) a perturbation subgraph; (d) another perturbation subgraph.
observability and redundancy classification in process networks

For the steady state process network under consideration if the constraints consist of mass balances,

\[ A_m x = 0 \]  

(2)

only, where \( x \) is the vector of mass flows and \( A_m \) is the incidence matrix of \( G_m \), the process network is termed a mass flow network. If the constraints contain the energy balances

\[ A y = 0 \]  

(3)

as well, where \( y \) is the vector of energy flows and \( A \) the incidence matrix of \( G \), it is termed a mass-energy flow network.

Now the solution \( (x, y) \) to eqns (2) and (3) is related to the fundamental cycle (mesh) flows \( x^0 \) and \( y^0 \) by

\[ x = C_m^T x^0 \]  

(4)

\[ y = C y^0 \]  

(5)

where \( C_m \) and \( C \) are the fundamental cycle matrices of \( G_m \) and \( G \), respectively.

It is extremely important to note that the space of feasible solutions \( (x, y) \) for a mass-energy flow network (abbreviated as \( G \)) is linear. The nonlinearity of the energy balances appears only when we attempt to calculate the enthalpy \( H = y/x \) or the temperature. Thus, if \( (x_1, y_1) \) and \( (x_2, y_2) \) are both solutions to eqns (2) and (3),

so are their linear combinations, although some of these solutions may be physically infeasible, e.g. solutions giving rise to negative absolute temperatures. In practice this problem will not arise since only solutions close to known feasible solutions will be considered.

It also follows from the linearity of the solution space that linear combinations of feasible solutions for two subgraphs \( G_1 \) and \( G_2 \) are feasible solutions for \( G_1 \cup G_2 \) and that the space of feasible solutions for \( G \) is spanned by the vectors representing the feasible solutions for all possible subgraphs. In particular, the feasible solution space for \( G \) is minimally spanned by the feasible solutions for the subgraphs which are the fundamental cycles of \( G \). The last observation gives rise to the following important definition and result:

**Definition.** Let \( (x^*, y^*) \) be a feasible solution in \( G \) and let \( \delta x, \delta y \) be a feasible solution for \( G_1 \), where \( G_1 \subset G \), \( \delta x \rightarrow 0 \) and \( \delta y \rightarrow 0 \). \( G_1 \) is a perturbation subgraph of \( G \) at \( (x^*, y^*) \), if the measurements on \( G \) cannot distinguish \( (x^*, y^*) \) from \( (x^* + \delta x, y^* + \delta y) \).

**Lemma 1.** In a process network with the underlying graph \( G \), an arc variable \( (x_i, y_i \) or \( T_i \) is locally un-

The lemma follows immediately from the definition and the fact that feasible unmeasurable perturbations can occur in a perturbation subgraph. It provides a basis for establishing sufficient conditions for observability and unobservability. However, the required theorems are sometimes more conveniently stated in terms of feasible unmeasurable perturbations which will be studied next. Figure 2 illustrates the subgraph \( G_{mn} \) and two perturbation subgraphs derived from a simple process network.

**OBSERVABILITY CLASSIFICATION IN MASS FLOW NETWORKS**

For mass flow networks the properties needed for observability classification are fairly obvious. However, it is worthwhile stating the principal results in order to bring out the contrast with the more complex situation of mass-energy flow networks. We shall omit the proofs which are given elsewhere[9].

**Theorem 1.** Let \( G \) be the underlying graph of a mass flow network, let \( v \) be a cut-node of \( G \) and let \( G_1 \) and \( G_2 \) be two subgraphs such that \( G_1 \cup G_2 = G = G_m \) and \( G_1 \cap G_2 = v \). Then

(a) No net mass flow can cross node \( v \) from \( G_1 \) to \( G_2 \) (or \( G_2 \) to \( G_1 \)).

(b) No net mass flow can enter or leave a biconnected component of \( G_m \).

(c) Nonzero mass flow in arc \( i \) is feasible if and only if arc \( i \) lies in a cycle in \( G_m \).

**Theorem 2.** In a mass flow network mass flow in an arc \( i \) is unobservable if and only if arc \( i \) lies in a cycle of unmeasured arcs.

An immediate consequence of Theorem 2 is the following corollary which follows from Theorem 8 of our previous paper[10] and was previously proven by a different technique[6].

**Corollary.** For constrained least-squares estimation in a mass flow network, the flow estimate in an unmeasured arc \( i \) is nonunique if and only if arc \( i \) lies in an unmeasured cycle.

Note that since the constraints and measurement equations are both linear for mass flow networks, there is no distinction between local and global observability.

**OBSERVABILITY CLASSIFICATION IN MASS-ENERGY FLOW NETWORKS**

In mass flow networks the unmeasured cycle criterion (Theorem 2) provided both necessary and sufficient conditions for unobservability and observability. In mass-energy flow networks there are two types of measurements (flow and temperature) and three types of variables (temperature, mass and energy flows). With this increase in complexity, it will no longer be possible to write down such simple graph-theoretical conditions. Instead, sufficient conditions for observability and very different sufficient conditions for unobservability will be presented. Furthermore, since the energy balances are nonlinear due to the products of flow rates and enthalpies, it will be necessary to distinguish global observability criteria from local observability criteria.

It is important to note that the sufficient conditions for
mass flow observability implied by Theorem 2 for a mass flow network still apply in a mass-energy flow network. That is, if every cycle in $G_m$ containing arc $i$ has at least one flow measurement, then $x_i$ is observable using only mass balances and mass flow measurements. Hence $x_i$ is observable in the mass-energy flow network which includes mass balances and flow measurements as well as energy balances and temperature measurements. However, if arc $i$ is in a cycle of $G_m$ in which all mass flows are unmeasured, it may still be possible to calculate these flows using temperature measurements and energy balances. In fact, the difficulty in classifying observability in a mass-energy flow network arises precisely because of the possibility that a cycle of mass flow arcs might not have any mass flow measurements at all.

The blending system shown in Fig. 3 demonstrates the points just made. The flow $x_3$ lies in no cycles in $G_m$ and hence it is observable just using the mass balance equations and flow measurements. However, $x_4$ and $x_5$ form a cycle with no mass flow measurements. Only through the use of temperature measurements and the energy balance can $x_4$ and $x_5$ be calculated. It will be seen that the possibility of performing this calculation depends on the values of $H_s$ and $H_t$.

We now present the basic theorems needed for observability classification. As in the case of the mass flow network, it will be necessary to determine which flows are feasible.

**Theorem 3.** Let $G$ be the underlying graph of a mass-energy flow network, let $v$ be a cut-node of $G$, and let $G_1$ and $G_2$ be the subgraphs such that $G_1 \cup G_2 = G$ and $G_1 \cap G_2 = v$. Then

(a) No net mass or energy flow can cross node $v$ from $G_1$ to $G_2$ (or $G_2$ to $G_1$)

(b) No net mass or energy flow can enter or leave a biconnected component of $G$.

(c) Nonzero mass flow in an arc $i$ is feasible if and only if arc $i$ lies in a cycle of $G_m$.

(d) Nonzero energy flow in an arc $i$ is feasible if and only if arc $i$ lies in a cycle of $G$.

The proof of Theorem 3 is very similar to that of Theorem 1 and is omitted here for the same reason.

**Theorem 4.** Let $G$ be the underlying graph of a mass-energy flow network and let mass flow arcs $i$ and $j$ form a cut-set of $G$. $T_j$ is globally observable if $T_i$ is measured.

**Proof.** Let $G_1$ and $G_2$ be the two subgraphs of $G$ after deleting arcs $i$ and $j$ and let

$$a_i = \begin{cases} 1, & \text{if arc } i \text{ is directed from } G_1 \text{ to } G_2 \\ -1, & \text{if arc } i \text{ is directed from } G_2 \text{ to } G_1 \end{cases}$$

$$a_j = \begin{cases} 1, & \text{if arc } j \text{ is directed from } G_1 \text{ to } G_2 \\ -1, & \text{if arc } j \text{ is directed from } G_2 \text{ to } G_1. \end{cases}$$

Then the mass and energy balances about $G_2$ are

$$a_i x_i + a_j x_j = 0$$

$$a_i H_i + a_j H_j = 0.$$  \(8\)

$$a_i x_i (H_i - H_j) = 0.$$  \(9\)

Multiplying eqn (8) by $H_i$ and subtracting it from eqn (9), we obtain

$$a_i x_i (H_i - H_j) = 0.$$  \(10\)

If $x_j \neq 0$, $H_j = H_{iu}$ and $H_{iu}$ cannot be perturbed without affecting the measurement $T_i$. If $x_j = 0$, then by definition $H_i = 0$ and its value is again fixed.

**Corollary.** Let $G$ be the underlying graph of a mass-energy flow network and let mass flow arcs, $i_1, i_2, \ldots$ and $i_k$ form a cut-set of $G$. $T_j$ is locally observable, if $T_{i_1}, T_{i_2}, \ldots T_{i_k}$ are measured and $H_{i_1} = H_{i_2} = \ldots = H_{i_k}$.

We shall omit the proof which follows an analogous development to that of Theorem 4 but illustrate the application with reference to Fig. 4. Theorem 4 shows that $T_{i_1}, T_{i_2}$ and $T_{i_3}$ in Fig. 4(a) are globally observable, and by the corollary, if $H_{i_1} = H_{i_2}$, $T_3$ is observable and $H_{i_2} = H_{i_1} = H_{i_2}$ in Fig. 4(b).

The two foregoing results deal with sufficient conditions for global and local observability. The next three

![Fig. 4. Process networks.](image-url)
Theorems stipulate the sufficient conditions for global and local unobservability.

Theorem 5. Let \( G \) be the underlying graph of a mass-energy flow network.

(a) if arc \( i \) is in a cycle of \( C \) of \( G_{\text{mu}} \), then \( x_i \), \( y_i \) and \( T_i \) are globally unobservable;

(b) if arc \( i \) is in a cycle \( C \) of \( G_{\text{mu}} \) with exactly one temperature measurement, then \( x_i \) and \( y_i \) are globally unobservable.

Proof. (a) For the cycle \( C \) in \( G_{\text{mu}} \) choose any arc \( k \), \( k \neq i \), in \( C \) and choose a spanning tree for \( G_{\text{mu}} \) for which arc \( k \) is not a branch. Consider first an arbitrary perturbation \( \delta x_k \) with \( T_k \) held constant. Then \( \delta y_k = H_k \delta x_k \) and from eqns (4) and (5) we have also perturbed all \( x_j \) and \( y_j \) in \( C \) by \( \delta x_k \) and \( \delta y_k \), respectively. Since there are no measurements in \( C \), these perturbations are clearly feasible and unmeasured, and since the argument does not depend on the values of \( x_i \), \( y_i \) or \( H_i \) in \( C \), \( x_i \) and \( y_i \) are globally unobservable.

Next consider an arbitrary perturbation \( \delta y_k \) with \( x_k \) held constant. From eqn (5) all \( y_j \) in \( C \) are perturbed by \( \delta y_k \) and \( H_j = (y_j \pm \delta y_k) \) with the sign determined by the direction of arc \( j \). In either case \( H_j \) is perturbed to a new feasible value and there are no measurements in \( C \) to detect the change. Hence, by the same argument \( T_i \) is globally unobservable.

(b) Now use the same construction as before but let arc \( k \) be the arc with the temperature measurement. The first type of perturbations \( (\delta x_k, \delta y_k) \) is still feasible and unmeasured, but the second type \( (\delta y_k, \delta T_k) \) would be detected. Hence \( x_k \) and \( y_k \) are globally unobservable, but no conclusion is drawn about the observability of \( T_k \).

An illustration of the above theorem may be found in Fig. 5(a). Arcs 5 and 6 form a cycle in \( G_{\text{mu}} \) with exactly one temperature measurement. Hence \( x_s \), \( x_n \), \( y_s \) and \( y_n \) are globally unobservable. Note that even though mass and energy flows are globally unobservable in a cycle of \( G_{\text{mu}} \) with exactly one temperature measurement, it is still

![Fig. 5](image-url)
possible for the temperatures to be locally or globally observable. In Fig. 4(b) \( x_2 - x_7, y_4 - y_7 \) are globally unobservable but \( T_3 \) is locally observable if \( H_s = H_{10} \) (Corollary to Theorem 4), and in Fig. 4(a) \( x_2, x_5, x_3, y_1, y_2, y_3 \) are globally unobservable but \( T_1, T_2, T_3 \) are globally observable.

**Theorem 6.** Let \( G \) be the underlying graph of mass-energy flow network and let arc \( i \) be in a cycle \( C \) in \( G \) with no temperature or energy flow measurements. Then \( y_i \) is globally unobservable. Furthermore, if arc \( i \) is a mass flow arc, then \( T_i \) is also globally unobservable.

**Proof.** Choose any arc \( k \), \( k \neq i \), in \( C \) and a spanning tree for \( G \) for which arc \( k \) is not a branch. Perturb \( y_k \) arbitrarily by \( \delta y_k \) while holding \( x_k \) constant. For each arc \( j \) in \( C \), \( y_j \) is perturbed by \( \delta y_k \) according to eqn (5). Furthermore, since \( H_j = y_j/x_j \) for mass flow arcs, \( H_j \) is also perturbed. Since no measurements detect these feasible perturbations, the theorem follows.

One difference between this theorem and the preceding one is that the cycle in Theorem 6 might contain mass flow measurements which are explicitly excluded in Theorem 5.

Suppose all mass flows are globally observable just using the mass flow measurements and mass balances. Then any temperature or energy flow can be classified as globally unobservable by Theorem 6 or globally observable by the following corollary:

**Corollary.** Let \( G \) be the underlying graph of a mass-energy flow network, let \( G_{ac} \) be acyclic, and let arc \( i \) be any arc in \( G \). If every cycle containing arc \( i \) has a temperature or energy flow measurement, then \( y_i \) is globally observable, and so is \( T_i \) if arc \( i \) is a mass flow arc. The proof of the corollary follows immediately from Theorems 2 and 3(d).

![Fig. 6. Erroneous classification.](image)
In many cases observability will depend on the temperature or enthalpy values of some arcs. The following local unobservability theorem is an extension of Theorem 5:

Theorem 7. Let $G$ be the underlying graph of a mass-energy flow network and let arc $i$ be an arc in a cycle $C$ in $G_{mu}$. If there is at most one value of enthalpy calculated from all temperature measurements on $C$, then $x_i$ and $y_i$ are locally unobservable. Furthermore, if $T_i$ is unmeasured, and if $H_i \neq H_i$, $T_i$ being a measured temperature, then $T_i$ is also locally unobservable.

Proof. Let arc k, $k \neq i$, be one of the arcs in C with a temperature measurement. Choose a spanning tree for $G_{mu}$ for which arc k is not a branch. Let $x^0$ and $y^0$ denote the mass and energy flows before the perturbations. Consider an arbitrary perturbation $\delta x_k$ with $T_i$ held constant. Then $x_k = x_k^0 + a \delta x_k$, $H_k = H_k^0$, and $y_k = y_k^0 + H_k^0 \delta x_k$, and for any other arc j in C, $x_j = x_j^0 + a \delta x_k$ and $y_j = y_j^0 + a \delta y_k = x_j^0 H_j^0 + a H_j^0 \delta x_k$, where $a = 1$ (or $-1$) if arc j is oriented in the same (or opposite) direction as arc k. But for all the arcs in C with measured temperature $H_j^0 = H_j^0$. Hence $y_j = x_j H_j^0$ and $H_j = H_j^0$. Thus, these temperature measurements would not detect the mass and energy flow perturbations, and $x_i$ and $y_i$ are locally unobservable.

Now on the unmeasured arc i in C, $H_i = y_i / x_i = (H_i^0 x_i^0 + a H_i^0 \delta x_k) / (x_i^0 + a \delta x_k)$. If $H_i^0 \neq H_i^0$, then the value of $H_i$ depends on $\delta x_k$, which is arbitrarily chosen. Hence, $T_i$ is also locally unobservable.

Some illustrations of Theorem 7 can now be given. If $H_4 = H_5$, arcs 4 and 5 in Fig. 3 form a cycle as stipulated in Theorem 7 and $x_3$, $y_4$, $x_5$, $y_5$ are locally unobservable. In Fig. 5(a), if $H_4 = H_5$, $x_4$, $x_5$, $x_3$, $x_7$, and $y_4$, $y_5$, $y_7$ are locally unobservable, and if $H_5 \neq H_4$, then $T_3$ is also locally unobservable. We note that Theorem 7 applies to cycles with more than one measurement and provides more information than Theorem 5 even for a mass flow cycle with a single temperature measurement. For instance, in the cycle formed by arcs 4, 5, 6 and 7 in Fig. 4(b), if $H_4 \neq H_5$, $T_3$ is locally unobservable by Theorem 7.

As we pointed out earlier, observability classification based on matrix rank tests can only yield information about the system as a whole, whereas from the viewpoint of process analysis it is often crucial to identify the individual variables with the various levels of observability or unobservability. The matrix rank tests are also computationally cumbersome. For the mass-energy flow network in Fig. 5 the rank of a 38 x 37 matrix would have to be tested. In this case the test is inconclusive. But in any case the matrix rank test can only establish local system observability or unobservability, since the evaluation must be carried out at a particular set of conditions. By comparison with the matrix rank tests which are basically numerical tests, the potency of graph-theoretical classification criteria becomes very apparent.

Because of these shortcomings matrix rank tests should be used only as a last resort after we have exhausted all the other alternatives of observability classification. In the algorithm to be presented the graph-theoretic theorems will be used to classify as many variables as possible. These variables can then be removed from further consideration. The elimination usually results in the decomposition of the process network into smaller sub-networks. With this reduction in dimensionality the matrix rank tests can then be applied effectively.

It is important to note that in the following theorem which gives both necessary and sufficient conditions for local observability we will only be concerned with subgraphs of the original process graph. The absence or existence of perturbation subgraphs will be the basis for local observability or unobservability.

It should be clear that an arc with both mass flow and temperature measurements would admit no unmeasurable perturbation in mass or energy flow. After its temperature, mass and energy flows have been classified as globally observable, it can be safely deleted from further considerations. Arcs 1, 12, 15, 16, 17 and 18 in Fig. 5(a) fall into this category. Similarly a pure energy arc i with an energy flow measurement would admit no unmeasurable perturbation in energy flow. It too can be deleted after $\gamma_i$ has been classified as globally observable.

Next consider a mass flow arc i with mass flow measurement but no temperature measurement. Clearly it would admit no unmeasured mass flow perturbation, but it could admit unmeasured energy flow perturbation through temperature changes. Since arc $i$ cannot be in the mass flow perturbation subgraph, it may be converted to a pure energy flow arc with the understanding that $T_i$ and $y_i$ of the original arc are observable if and only if $y_i$ in the corresponding pure energy flow arc is observable. Such an arc should be so converted after its mass flow has been classified as globally observable. Arc 10 in Fig. 5(a) is such an example.

It should be evident that after these deletions and conversions, the only arcs remaining will be mass flow arcs with measured temperatures only, unmeasured mass flow arcs, and unmeasured pure energy flow arcs. Let the incidence matrices correspond to these arcs be $A_{mu}$, $A_{mu}$, and $A_{nu}$, respectively, and define a new matrix $B_1$ by

$$B_1 = A_{m1} / A$$

where $A$ is a diagonal matrix whose elements are the enthalpies $H_j$ corresponding to the arcs in $A_{m1}$. The most general case of a matrix to be rank-tested will contain sub-matrices $A_{m1}$, $A_{m2}$, $A_{nu}$, and $B_1$.

Theorem 8. Let a mass-energy flow network contain only mass flow arcs with measured temperatures, unmeasured mass flow arcs, and unmeasured pure energy flow arcs, let the incidence matrices corresponding to these arcs be $A_{m1}$, $A_{nu}$ and $A_{nu}$, respectively, and let the energy flow coefficient matrix corresponding to $A_{nu}$ be $B_1$. The network is locally observable if and only if the rank of the partitioned matrix

$$\begin{pmatrix} A_{m1} & A_{m2} & 0 & 0 \\ B_1 & 0 & A_{nu} & A_{nu} \end{pmatrix}$$

is equal to $n$, the total number of mass and energy flow arcs.
Proof. Let $\delta x^i$ be the mass flow perturbations in (the subgraph corresponding to) $A_{m_1}$, which give rise to corresponding energy flow perturbations in $B_1$, let $\delta x^2$ and $\delta y^2$ be the mass and energy flow perturbations in $A_{m_2}$, and let $\delta y^3$ be the energy flow perturbations in $A_e$. Since these are the only unmeasurable perturbations and since the nodal mass and energy balances are given by

$$
\begin{pmatrix}
A_{m_1} & A_{m_2} & 0 & 0 \\
B_1 & 0 & A_e & 0
\end{pmatrix}
\begin{pmatrix}
\delta x^1 \\
\delta x^2 \\
\delta y^2 \\
\delta y^3
\end{pmatrix}
= 0
$$

(13)

any nonzero solution to eqn (13) would constitute a feasible unmeasurable perturbation. But the theorem stipulates the necessary and sufficient condition for precluding such nonzero solutions, and by Lemma 1 the network is locally observable if and only if it contains no perturbation subgraphs.

Note that the rank of the partitioned matrix depends on the enthalpy values used in eqn (11). However, if the number of rows of matrix (12) is less than $n$, the system is unobservable for any enthalpy values.

**Observability Classification Algorithm**

Starting with a process network with its arc attributes of measured variables and enthalpy values, the algorithm classifies each temperature, mass and energy flow into one of the following categories: globally observable (g.o.), locally observable (I.o.), locally unobservable (I.u.), globally unobservable (g.u.) or an “unobservable block” (u.b.). The last category indicates that the arc is in a perturbation subgraph which cannot be further classified. An unobservable block contains at least one locally unobservable variable.

The basic approach taken in the algorithm is to determine which arcs are in which perturbation subgraphs of $G$. Initially the entire process graph is considered as a candidate for perturbations. If an arc cannot possibly be in any perturbation subgraph, it is deleted. If a perturbation subgraph can be identified, it is aggregated into a pseudo-node which is aggregated in the unclassified sub-network of the original process network. At any time during the execution of the algorithm the network under consideration is the unclassified sub-network of the original process network. In this connection it is important to note that the identities of $G$, $G_m$ etc. can change during the execution of the algorithm, referring always to the unclassified subgraphs remaining at any instant.

We shall now state the algorithm and then illustrate the various steps with respect to the network in Fig. 5:

1. **Arc reduction rules**
   For each arc $i$ in $G$ call REDUCE (i) which is defined below.

   Procedure REDUCE (i):
   A. For a mass flow arc $i$
      i. classify $T_i$ as g.o., if $T_i$ is measured.
      ii. classify $x_i$ as g.o., if $x_i$ is measured. Furthermore,
      a. delete arc $i$ and classify $y_i$ as g.o., if $T_i$ is also measured.
   b. convert arc $i$ to an unmeasured pure energy flow arc, if $T_i$ is not measured. The observability of $T_i$ will be the same as that of $y_i$.
   B. For a pure energy flow arc $i$, delete arc $i$ and classify $y_i(T_i)$ as g.o., if $y_i$ is measured.
   C. Exceptions: Variables which have been previously classified should not be reclassified.

2. **Unobservable cycle rules**
   A. Find components and biconnected components of $G_{max}$.
   B. Aggregate all nodes in each biconnected component of $G_{max}$ with more than one arc. All unmeasured variables in the arcs so eliminated are g.u.
   C. For each pure energy flow arc $i$ with incident nodes $u$ and $w$ in the same component of $G_{max}$, $y_i$ is g.u.
      i. For all arcs $j$ in the cycle formed by arc $i$ and arcs in $G_{max}$ after $y_i$ and $T_i$ are classified g.u., delete arc $i$.
   D. For each arc $k$ in $G_m$ with a measured temperature and with incident nodes $v$ and $w$ in the same component of $G_{max}$ aggregate nodes $v$ and $w$ and also aggregate all nodes in the cycle formed by arc $k$ and arcs in $G_{max}$. For each arc $i$ in $G_{max}$ so eliminated do not reclassify $T_i$ if already classified, but
      i. classify $T_i$ as l.u., if $H_i \neq H_o$.
      ii. classify $T_i$ as in an u.b., if $H_i = H_o$.
   All other unmeasured variables in the eliminated arcs are g.u.

3. **Feasibility rules**
   A. Mass flow feasibility
      i. Find biconnected components of $G_m$.
      ii. For each biconnected component of $G_m$ with exactly one arc $i$ mark $x_i$ as measured and call REDUCE (i).
   B. Energy flow feasibility
      i. Find components, biconnected components and cut-nodes of $G$.
      ii. For each biconnected component of $G$ with exactly one arc $i$ mark $y_i, T_i$ and also $x_i$ as measured, and call REDUCE (i).
   C. Perform splitting operation on each node $v$ which is still a cut-node of $G$.

4. **Parallel arc rule**
   If arcs $i$ and $j$ are parallel pure energy flow arcs, delete arc $j$ and classify $y_i$ and $y_j$ as g.u.

5. **Matrix rules**
   Aggregate the nodes in each component of $G_{max}$.
   For each component of $G$ apply matrix rank test. Classify each previously unclassified variable as
   A. l.o. if the component is l.o.
   B. g.u. if the matrix has fewer rows than columns.
   C. in an u.b. if the component is l.u.

**AN EXAMPLE**

We shall now apply the observability classification algorithm step by step to the process network in Fig. 5(a).
1. Application of arc reduction rules results in the deletion of arcs 1, 12, 15, 16, 17 and 18 after their temperatures, mass and energy flows have been classified as g.o., and in the conversion of mass flow arc 10 to a pure energy flow arc after $x_{10}$ has been classified as g.o. The revised $G$ is shown in Fig. 5(b).

2. The unmeasured cycle rules are based on the application of Theorems 5, 6 and 7. Step 2B aggregates the nodes and classifies the temperatures, mass and energy flows as g.o. in a completely unmeasured cycle of mass flow arcs according to Theorem 5(a). The reason for aggregation is to reduce the size of the network to be further classified. The variables on all eliminated arcs have been classified, and consequently, require no further attention. Since aggregation of two nodes is algebraically equivalent to combining two rows in the mass balance equations (2) and combining two rows in the energy balance equations (3), the solution in terms of variables external to the pseudo-node are unaffected.

In the process of aggregating nodes $v$ and $w$ any mass flow arc $k$ with measured temperature connecting the same two nodes will also be eliminated. Since there is at least one path between nodes $v$ and $w$ in $G_{mm}$ arc $k$ must be in a cycle with exactly one temperature measurement. By Theorem 5(b) $x_v$ and $y_v$ are g.u. An unmeasured pure energy flow arc $k$ between nodes $v$ and $w$ will similarly be eliminated, and by Theorem 6, $y_v$ is also g.u. All these situations are accounted for in step 2B, but since they do not arise in Fig. 5(b) the network is unaltered in this step.

After step 2B $G_{mm}$ consists only of trees. In step 2C we consider each completely unmeasured cycle with exactly one pure energy flow arc. Then by Theorem 6 $y_v$ is g.u., and for each of the other arcs in the cycle (say, arc $j$ in $G_{mm}$) $T_j$ and $y_j$ are also g.u. In this step the adjacent nodes $v$ and $w$ are not aggregated since the mass flows in the cycle have not yet been classified. Instead arc $k$ is deleted. This deletion will not affect potential unmeasurable energy flow perturbations in arcs external to the cycle. Again step 2C does not affect the network in Fig. 5(b).

In step 2D we consider each remaining cycle in $G_{m}$ with exactly one temperature measurement $T_i$. For each arc $i$ in the cycle $x_i$ and $y_i$ are g.u. by Theorem 5(b). Furthermore, by Theorem 7 $T_i$ is l.u. if $H_i \neq H_o$. $T_i$ is in an u.b. if $H_i = H_o$ since arc $i$ is in a perturbation subgraph. In either case all nodes in the cycle are aggregated into a pseudo-node with no effect on potential unmeasurable perturbations in arcs external to the pseudo-node. Note that if $T_i$ has already been classified as g.u. in step 2C, it should not be reclassified. In the example in Fig. 5(b) nodes 3 and 4 are aggregated. $x_5$, $x_6$ and $y_5$, $y_6$ are g.u. $T_5$ has been previously classified as g.o. in step 1. If $H_5 \neq H_o$, $T_5$ is l.u., and if $H_5 = H_o$, $T_5$ is in an u.b.

3. In steps 3A and 3B infeasible mass and energy flow perturbations are identified using feasibility rules based on Theorems 1 and 3. Arcs which do not permit unmeasurable perturbations are tagged and reprocessed by arc reduction rules. We begin with each biconnected component with exactly one arc $i$ in the mass flow sub-network. Since mass flow perturbation in such an arc is impossible by Theorem 1(c), $x_i$ is treated as if it were measured. By tagging it as “measured” we can immediately subject it to reprocessing by arc reduction rules without any other special provisions.

Similarly, unmeasurable mass and energy flow perturbations are ruled out in a biconnected component of $G$ containing exactly one arc $i$ by Theorem 3(c) and (d). To avoid the inconvenience of distinguishing whether arc $i$ is a mass or pure energy flow arc, mark $T_i$, $y_i$ and $x_i$ as “measured” and invoke the arc reduction rules. Notice that $x_i$ is tagged in this case for convenience.

Applying step 3A to the network in Fig. 7(c) we mark $x_2$, $x_{11}$, $x_{13}$ and $x_{19}$ as “measured” and after appropriate arc reductions the network appears as shown in Fig. 7(d).

Similarly, $y_2$, $T_2$ and $y_{14}$ and $T_{14}$ are tagged as “measured” in step 3B which gives rise to the network in Fig. 7(e). In step 3C we perform splitting on node 3, the cut-node in $G$. The splitting permits the matrix rank test to be applied to disjoint subgraphs of lower dimensions shown in Fig. 7(f).

At this point the need for avoiding reclassification in step 1C may be illustrated with a simple example. Referring to Fig. 6(a), $y_1$, $y_2$ and $T_1$ are classified as g.u. in step 2C. After the deletion of arc 2, arc 1 is not in any cycle. Hence $x_1$ is g.o. However, if we do not prohibit reclassification, it might be erroneously concluded from Fig. 7(b) that $T_{11}$ and $y_1$ are also g.o.

4. The parallel arc rule applies to unmeasured pure energy flow arcs whose flows may be classified as g.u. by Theorem 6. All but one of the energy flow arcs may be deleted after classification. Notice that in this step the adjacent nodes $v$ and $w$ must not be aggregated. Since the arcs between $v$ and $w$ can only transmit pure energy flow, aggregation would introduce a mass flow path which does exist in the unaggregated network. For example, Fig. 7(b) would suggest feasible unmeasurable mass flow perturbations through arcs 1 and 2, which are not permitted in Fig. 7(a).

5. At this point we have exhausted all the graph-related reduction and classification rules. A comparison of Fig. 5(a) and 5(f) shows clearly the dramatic reduction in network dimensionality. However, we can make a further reduction by aggregating the nodes in each component of $G_{mm}$ before applying the matrix rank test. Since a completely unmeasured path of mass flow arcs connect these nodes, the aggregation clearly does not
affect feasible unmeasurable perturbations external to these nodes. If the matrix rank test shows a component of G containing such a pseudo-node to be l.o., then the variables associated with the eliminated arcs in the pseudo-node are also l.o. If the test shows the component to be l.u. we can only classify the eliminated arcs as being in an u.b.

With reference to the example, aggregation of nodes 5 and 6 transforms Fig. 5(f) to Fig. 5(g), and the application of Theorem 8 shows that arcs 8–11 are in a g.u. block and that arcs 3, 4 and 7 are l.o. if $H_r \neq H_e$ and in a l.u. block if $H_r = H_e$.

**REdundancy Classification in Mass Flow Networks**

Since redundancy is defined in terms of observability[10], the redundancy of a measured variable can be determined by deleting the measurement and applying the observability classification algorithm. Depending on whether the unmeasured variable is globally (locally) observable or unobservable, the measurement is globally (locally) redundant or non-redundant. This relationship is exploited in proving redundancy theorems, but as a computational procedure, it is not very efficient and should be avoided except as a last resort. In fact much of the redundancy classification algorithm is devoted to strategies of classification without resorting to this “brute-force” procedure. It is important to note that, unlike observability, redundancy is defined only with respect to measured variables, although they both depend on network configuration and measurement placements.

As before, we shall first establish the governing theorems, then present the classification algorithm and finally illustrate the steps in the algorithm with an example.

Since a mass flow network is linear, its properties with respect to observability and redundancy are global. We need simply to characterize a measurement as redundant or non-redundant. The following theorem states three equivalent criteria for redundancy.

**Theorem 9.** In a mass flow network let arc i connecting nodes v and w be measured. Then the flow measurement $x_i$ is redundant if and only if

(a) nodes v and w are not connected by a path of unmeasured arcs (regardless of directions);

(b) nodes n and w lie in different components of $G_{nu}$;

(c) arc i is in a cut-set of $G_{nu}$ consisting solely of measured arcs.

**Proof.** (a) To test for non-redundancy delete the measurement at arc i. The deleted measurement was non-redundant if and only if $x_i$ is now unobservable, which is true if and only if arc i now lies in a cycle of unmeasured arcs (Theorem 2), which is true if and only if nodes v and w are connected by a path of unmeasured arcs other than arc i. Hence the measurement $x_i$ is redundant, if and only if nodes v and w are not connected by a path of unmeasured arcs.

(b) If nodes v and w lie in two different components of $G_{nu}$, they cannot be connected by a path of unmeasured arcs. Hence the result follows directly from (a).

(c) This is equivalent to case (b).

To illustrate the application of the above theorem let us consider Fig. 8. Since no measured arc in Fig. 8 lies in a cycle with only one measurement, all measurements, $x_i - x_6$, are redundant. Yet the system as a whole is unobservable because of the cycle of unmeasured arcs 8–10. If we switch the measurement from arc 5 to arc 8, the system becomes observable and the measurements $x_4$, $x_6$ and $x_8$, each lying in a cycle with exactly one measurement, become non-redundant.

**Redundancy Classification in Mass-Energy Flow Networks**

**Theorem 10.** In a mass-energy flow network let nodes v and w be connected by arc i. Then

(a) if arc i is a mass flow arc and if v and w are connected by a path in $G_{nu}$ the mass flow measurement $x_i$ is globally non-redundant;

(b) if v and w are connected by a path with no temperature or energy flow measurements, the temperature measurement $T_i$ or energy flow measurement $y_i$ is globally non-redundant.

**Proof.** (a) Delete the measurement. By Theorem 5 $x_i$ is now globally unobservable, and hence the measurement was non-redundant.

(b) Delete the temperature or energy flow measurement. By Theorem 6 $T_i$ or $y_i$ is now globally unobservable, and hence the measurement was non-redundant.

As an illustration of this theorem, let us consider the network in Fig. 9(a). By Theorem 10 (b) the temperature measurements $T_9$, $T_9$ and $T_9$ are all globally non-redundant, because they each lie in a cycle (arcs 5, 6, arcs 5, 8, 10, 11, and arcs 9, 10, 11, respectively), with exactly one temperature measurement. Notice that we could not characterize the redundancy of any of the flow measurements on the basis of this theorem, but the next theorem which give sufficient conditions for redundancy of a mass flow measurement in a mass-energy flow network will permit further classification of some flow measurements.

**Theorem 11.** In a mass-energy flow network let the mass flow of arc i connecting nodes v and w be measured. The measurement $x_i$ is globally redundant if (a) v and w are in different components of $G_{nu}$ or (b) arc i is in a cut-set of $G_{nu}$ consisting solely of mass flow arcs with flow measurements.

**Proof.** This theorem is a simple extension of Theorem 9. The conditions stipulated are sufficient to prove the
Fig. 9. Illustration of the steps in the redundancy classification algorithm.
redundancy of the flow measurement \( x_i \) using mass conservation relations only. Notice that the theorem makes no stipulation on temperature measurements.

By this theorem the measurements \( x_{15} - x_{14} \) in Fig. 9(a) are all globally redundant. The next theorem will give sufficient conditions for redundancy of an energy flow measurement in a mass-energy flow network.

**Theorem 12.** In a mass-energy flow network let a pure energy flow arc \( i \) connecting nodes \( v \) and \( w \) be measured. The measurement \( y_i \) is globally redundant if (a) \( v \) and \( w \) are in different components of \( G_s \); or (b) arc \( i \) is in a cut-set of \( G \), consisting solely of measured pure energy flow arcs.

**Proof.** If arc \( i \) is the only arc connecting two components of \( G_s \), as in (a), any flow or perturbation is clearly infeasible even without the measurement. Hence \( y_i \) is observable and the measurement on \( y_i \) is redundant. Case (b) is equivalent to case (a).

Applications of Theorem 12 often arise as a result of converting mass-energy flow arcs to pure energy flow arcs during the execution of the redundancy classification algorithm which is presented next.

**REDUNDANCY CLASSIFICATION ALGORITHM**

The redundancy classification algorithm requires the same input data as the observability classification algorithm. It classifies each measurement as globally redundant (g.r.), globally non-redundant (g.n.), locally redundant (l.r.), locally non-redundant (l.n.), or as in a locally "non-redundant" block (l.n.b). The l.n.b. category indicates that the measured variable is in a subgraph which is locally unobservable without the measurement, but that a further classification of the subgraph in terms of redundant and non-redundant measurements is not possible. The l.n.b. classification arises when a variable is classified as in a u.b. by the observability algorithm after the deletion of the measurement.

The general approach taken in the redundancy classification algorithm resembles that of the observability classification algorithm. Aggregation is performed when it does not affect the future classification of arcs external to the resulting pseudo-node and after all measurements internal to the pseudo-node have been classified. An arc is deleted from a process graph or a derived graph after it has been shown that no feasible unmeasurable perturbations can occur in that arc and that all variables associated with that arc can be calculated using only redundant measurements. No measurement is deleted until it has been appropriately classified. After we have exhausted all the measurements which can be directly classified using the redundancy classification theorems, the definition of redundancy is invoked. The measurement is temporarily deleted and the observability classification algorithm is applied to the residual subgraphs.

As with the observability algorithm, at any stage of algorithm execution the network under consideration is the unclassified sub-network of the original process network, with the identities of \( G, G_s \), etc. continually undergoing metamorphosis.

We shall now present the algorithm.

1. **Aggregation**

   For each completely unmeasured arc \((s, w)\) in \(G_m\) aggregate nodes \(s\) and \(w\). Classify as g.n. any measured variables in the arcs so eliminated.

2. **Redundancy in mass flow cut-set**

   A. Find components of \(G_m\).
   B. For each arc \((v, w)\) with a mass flow measurement \(x_i\) if nodes \(v\) and \(w\) lie in two components of \(G_m\):
      i. classify the measurement \(x_i\) as g.r.
      ii. convert arc \(i\) to an unmeasured pure energy flow arc, if \(T_i\) is unmeasured, or a measured pure energy flow arc, if \(T_i\) is measured. The classification of the \(T_i\) measurement will be the same as that of \(y_i\) measurement.

3. **Mass flow feasibility**

   A. Find biconnected components of \(G_m\).
   B. For each biconnected component of \(G_m\) with exactly one arc \(i\), convert arc \(i\) to a measured pure energy flow arc. The classification of the \(T_i\) measurement will be the same as that of the \(y_i\) measurement.

4. **Redundancy in energy flow cut-set**

   A. Find components of \(G_s\).
   B. For each pure energy flow arc \((v, w)\) with measurement \(y_i\) if nodes \(v\) and \(w\) lie in two components of \(G_s\), classify the energy flow measurement \(y_i\) as g.r. and delete arc \(i\).

5. **Energy flow feasibility**

   A. Find biconnected components, cut-nodes and components of \(G\).
   B. For each biconnected component of \(G\) with exactly one arc \(i\), classify every measurement on arc \(i\) as g.r. and delete arc \(i\).
   C. Perform splitting operations on each cut-node \(v\) of \(G\) to separate its biconnected components.

6. **Parallel arc rules**

   A. For each mass flow arc \(i\) with a temperature measurement, if it is parallel to an unmeasured pure energy flow arc or an arc in \(G_m\) with a mass flow measurement only, classify the measurement \(T_i\) as g.n.
   B. For each arc \(i\) with mass flow measurement only, if it is parallel to an arc in \(G_m\), classify the measurement \(x_i\) as g.n.

7. **"Brute force" rules**

   For each component of \(G\) derived in step 5C temporarily delete each unclassified measurement in turn and apply the observability classification algorithm to determine its status as follows:

<table>
<thead>
<tr>
<th>Variable</th>
<th>Classify the Measurement</th>
</tr>
</thead>
<tbody>
<tr>
<td>g.o.</td>
<td>g.r.</td>
</tr>
<tr>
<td>l.o.</td>
<td>l.r.</td>
</tr>
<tr>
<td>g.u.</td>
<td>g.n.</td>
</tr>
<tr>
<td>l.u.</td>
<td>l.n.</td>
</tr>
<tr>
<td>u.b.</td>
<td>l.n.b.</td>
</tr>
</tbody>
</table>
AN EXAMPLE

We shall now illustrate the step-by-step application of the redundancy classification algorithm to the process network in Fig. 9(a).

1. The aggregation in step 1 is a direct application of Theorem 10. This step results in the elimination of arcs 2, 3, 5, 6 and 11 and the revised network as shown in Fig. 9(b). The measurement $T_6$ is classified g.r.

2. The classification of mass flow measurement $x_i$ in step 2 is based on the conditions stipulated in Theorem 11. Since it is not yet determined if energy flow perturbations in arc $i$ are detectable by redundant measurements, arc $i$ is not deleted but converted to a pure energy flow arc. Because the mass flow is measured and known, the measurement $T_i$ is equivalent to the measurement $y_i$ and treated equivalently in the classification. It is not deleted because it is not yet classified. The network at the completion of this step is shown in Fig. 9(c), and the measurements $x_{12}, x_{13}, x_{17}, x_{18}$ are classified g.r.

3. After step 2 there may be mass flow arcs left which are no longer in cycles of $G_m$. These arcs are converted to pure energy flow arcs in step 3, since non-zero mass flows in these arcs are ruled out by Theorem 3(c). Arc 19 in Fig. 9(c) is such an example. Since only redundant measurements were deleted from the network in step 2, the implication is that $x_{19}$ is calculated using only redundant flow measurements. Note that if arc 19 had a mass flow measurement, it would have belonged to a cut-set in step 2 and eliminated before step 3. Similarly it would have been eliminated in step 1 if it were completely unmeasured. Hence the acyclic arcs of $G_m$, such as arc 19, must have exactly one temperature measurement each. As in step 2, the temperature measurement is treated equivalently as an energy flow measurement and retained for further processing. The outcome of step 3 is shown in Fig. 9(d).

4. In this step Theorem 12 is applied to the classification of energy flow measurements. Measurements satisfying the sufficient conditions of Theorem 12 are classified as g.r. and the associated arcs can be eliminated from further consideration after the classification. Figure 9(e) shows the outcome of this step. The measurements $T_{17}, T_{19}$ are classified as g.r.

5. In this step we invoke Theorem 3(d) to eliminate arcs with infeasible perturbations or zero flows. By virtue of the prior processing steps the only acyclic arcs remaining must be energy flow arcs with or without temperature (energy flow) measurements. Such arcs can be eliminated from further considerations after the associated temperature measurements have been classified as g.r. After such arcs, if any, have been eliminated the network is split according to its biconnected components to reduce the dimensionality and computing time in the subsequent processing steps. Since there are no acyclic arcs in Fig. 9(e), we proceed directly to Fig. 9(f).

6. The temperature measurement classification is a direct application of Theorem 10(b), and the mass flow measurement classification is based on Theorem 5(b). In contrast to the earlier steps no node aggregation or arc deletion is performed. In this step the measurements $T_8$ and $T_9$ in Fig. 9(f) are classified as g.n.

7. At this point the definition of redundancy is invoked to classify the remaining unclassified measurements. A measurement is redundant if and only if its deletion causes no loss in observability in the system. Each measurement is deleted in turn, and its observability is determined by the application of the observability classification algorithm. On that basis the measurements (a) $x_i$ is in a l.n. block if and only if $H_i = H_d$; (b) $y_i$ is in a l.n. block if and only if $H_i = H_d$; (c) $T_i$, $T_4$, $T_7$, $T_{12}$ and $T_{13}$ are g.r.

CLOSING REMARKS

In this paper we introduce the concepts of biconnected components, perturbation subgraphs and feasible unmeasurable perturbations and show how their properties may be used to effect observability and redundancy classification, simplification and dimensional reduction. These powerful graph-theoretic techniques are embodied in two classification algorithms for observability and redundancy in process networks. The treatment which assumes only mass and energy conservation constraints may be readily extended to process networks with other additional constraints, e.g. specified split fractions. It can be shown[9] that most of the graph-theoretic results may be generalized, the exceptions being Theorems 2, 6, 7 and 8, and the sufficient conditions of Theorems 1(c), 3(c) and 3(d).

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NOTATION

- $A$ the incidence matrix of graph $G$
- $A_m$ the incidence matrix of graph $G_m$
- $A_{m_1}$ the incidence matrix of a sub-network of $G_m$ with measured temperatures only
- $A_{m_2}$ the incidence matrix of a sub-network of unmeasured mass flow arcs
- $A_e$ the incidence matrix of a sub-network of unmeasured pure energy flow arcs
- $B_1$ an energy flow coefficient matrix defined by eqn (11)
- $C_1$ a fundamental cycle matrix of graph $G_m$
- $C'$ a fundamental cycle matrix of graph $G$
- $G$ a process graph
- $H$ a subgraph of $G$
- $G_i$ a subgraph of $G$
- $G_{m_i}$ the subgraph of $G$ with all measured energy flow arcs deleted
- $G_{m_2}$ the subgraph of $G_m$ with the deletion of all arcs with flow measurements
- $G_{m_3}$ the subgraph of $G_m$ with the deletion of all arcs with temperature measurements
- $H_i$ the enthalpy of material stream in arc $i$
- $T_i$ the temperature of material stream in arc $i$
- $v$ a node
- $x_i$ the mass flow in arc $i$
- $y_i$ a vector of mass flows
- $\delta x_i$ perturbation of vector of mass flows
- $x_i^*$ a vector of feasible mass flows in $G$
- $y_i$ the energy flow in arc $i$
- $y_i$ a vector of energy flows
- $\delta y_i$ perturbation of vector of energy flows
- $y^*$ a vector of feasible energy flows in $G$
- $w$ a node
A diagonal matrix whose elements are the enthalpies $H_i$
corresponding to the arcs in $A_\omega$.

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